

# A NOTE ON SPARING NUMBER ALGORITHM OF GRAPHS

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## Abstract

Let  $X$  denote a set of all non-negative integers and  $\mathcal{P}(X)$  be its power set. A weak integer additive set-labeling (WIASL) of a graph  $G$  is an injective set-valued function  $f : V(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$  where induced function  $f^+ : E(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$  is defined by  $f^+(uv) = f(u) + f(v)$  such that either  $|f^+(uv)| = |f(u)|$  or  $|f^+(uv)| = |f(v)|$ , where  $f(u) + f(v)$  is the sumset of  $f(u)$  and  $f(v)$ . The sparing number of a WIASL-graph  $G$  is the minimum required number of edges in  $G$  having singleton set-labels. In this paper, we discuss an algorithm for finding the sparing number of arbitrary graphs.

**Key words:** Integer additive set-labeled graphs, weak integer additive set-labeled graphs, sparing number of a graph, spring number algorithm.

**AMS Subject Classification:** 05C78.

## 1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [3, 5, 12]. Unless mentioned otherwise, all graphs considered here are simple, finite, non-trivial and connected.

The *sumset* of two sets  $A$  and  $B$  of integers, denoted by  $A + B$ , is defined as  $A + B = \{a + b : a \in A, b \in B\}$ . If  $A$  or  $B$  is countably infinite, then their sumset

$A + B$  will also be countably infinite. Hence, all sets we consider here are finite sets of non-negative integers.

Let  $X$  be a non-empty finite set of non-negative integers and let  $\mathcal{P}(X)$  be its power set. An *integer additive set-labeling* (IASL) of a graph  $G$  (see [4, 7]) is an injective function  $f : V(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$  such that the induced function  $f^+ : E(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$  is defined by  $f^+uv = f(u) + f(v) \forall uv \in E(G)$ . A graph  $G$  which admits an IASL is called an *integer additive set-labeled graph* (IASL-graph).

The cardinality of the set-label of an element (vertex or edge) of a graph  $G$  is called the *set-indexing number* of that element. An element of a given graph  $G$  is said to be a *mono-indexed element* of  $G$  if its set-indexing number is 1.

A *weak integer additive set-labeling* of a graph  $G$  is an IASL  $f : V(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$ , where induced function  $f^+ : E(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$  is defined by  $f^+(uv) = f(u) + f(v)$  such that either  $|f^+(uv)| = |f(u)|$  or  $|f^+(uv)| = |f(v)|$ , where  $f(u) + f(v)$  is the sumset of  $f(u)$  and  $f(v)$ .

**Lemma 1.1.** [9] *An IASI  $f : V(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$  of a given graph  $G$  is a weak IASI of  $G$  if and only if at least one end vertex of every edge of  $G$  mono-indexed.*

Hence, it can be seen that both end vertices of some edges of a given graph can be (must be) mono-indexed and hence those edges are also mono-indexed. The minimum number of mono-indexed edges required in a graph  $G$  so that  $G$  admits a WIASL is called the *sparing number* of  $G$ , denoted by  $\varphi(G)$ .

Note that an independence set  $I$  is said to have maximal incidence in  $G$  if maximum number of edges in  $G$  have their one end vertex in  $I$ . Then, the sparing number of any given graph can be determined using the following theorem.

**Theorem 1.2.** [8] *Let  $G$  be a given WIASL-graph and  $I$  be an independent set in  $G$  which has the maximal incidence in  $G$ . Then, the sparing number of  $G$  is the  $|E(G - I)|$ .*

Certain studies on WIASL-graphs and their sparing numbers have been done in [4, 7, 8, 9, 10]. In this paper, we discuss an algorithm to determine the sparing number of arbitrary finite connected graphs.

## 2 Sparing Number Algorithm

In this section, we use the following notations.

1.  $X :=$  The ground set used for labeling the elements of the graph  $G$ .
2.  $N(v) :=$  The set of all vertices in  $G$  adjacent to the vertex  $v$ .
3.  $N[v] := N(v) \cup \{v\}$ .

We now consider describe an algorithm to iteratively find out the sparing number of a given graph as explained below.

## 2.1 The Sparing Number Algorithm

- (1) Set  $G_1 = G$ ,  $X_1 = \emptyset$ ,  $Y_1 = \emptyset$ ,  $E_1 = \emptyset$ .
- (2) Choose  $v_i$  such that  $d(v_i) = \Delta(G_i)$ .
- (3) Label the vertex  $v_i$  by a non-empty, non-singleton subset of the ground set  $X$ .
- (4) Let  $G_{i+1} = G_i - \{v_i\}$ ,  $X_{i+1} = X_i \cup \{v_i\}$  and  $Y_{i+1} = Y_i \cup N(v_i)$ .
- (5) For any two vertices  $v_r, v_s \in N(v_i)$ , if  $v_r v_s \in E(G)$ , then let  $E_{i+1} = E_i \cup \{v_r v_s\}$ . If no two vertices in  $N(v_i)$  are mutually adjacent, then  $E_{i+1} = E_i$ .
- (6) Label the vertices in  $Y_{i+1}$  by distinct singleton subsets of the ground set  $X$ .
- (7) If all vertices of  $G_i$  are labeled, then go to step (8). Otherwise, go to step (2).
- (8) Here, the sparing number of  $G$  is  $\varphi(G) = |E_i|$ . Stop.

Note that the set  $I = \bigcup_i X_i$  is the independent set of vertices in  $G$  with maximal incidence in  $G$  and hence by Theorem 1.2, this algorithm provides the sparing number of any given graph. Also, the vertices of  $G$  in  $I$  have non-singleton set-labels whereas the vertices in  $V - I$  are mono-indexed.

## 2.2 An Illustrations to Sparing Number Algorithm

Consider the graph given below in Figure 1 for finding out the sparing number. Let  $X$  be a set of non-negative integers, which is used as the ground set for labeling the vertices of  $G$ . Let  $\mathcal{S}$  be the collection of all singleton subsets of  $X$  and  $\mathcal{X}$  be the collection of all non-empty, non-singleton subsets of the ground set  $X$ .

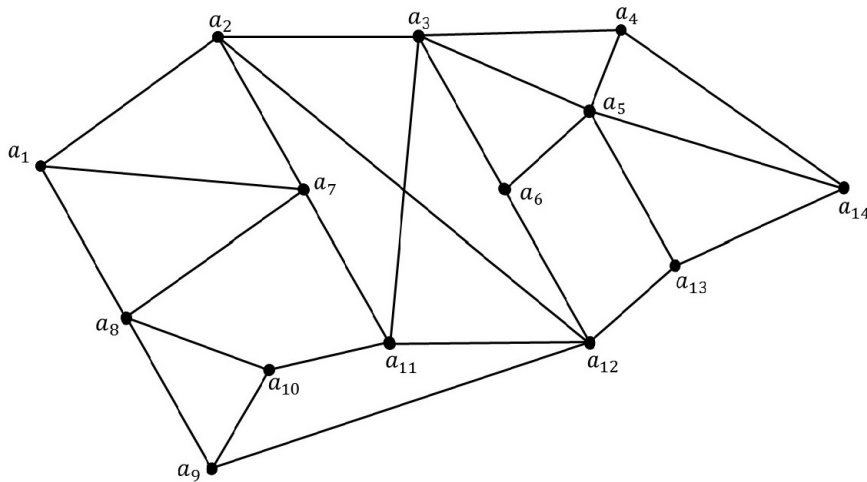


Figure 1

First, let  $G_1 = G$ ,  $X_1 = \emptyset$ ,  $Y_1 = \emptyset$  and  $E_1 = \emptyset$ . At this stage, we have  $\Delta(G_1) = 5$  and the vertices  $a_3, a_5, a_{12}$  have degree 5 in  $G_1$ . Without loss of generality, let  $v_1 = a_3$  and  $N(v_1) = \{a_2, a_4, a_5, a_6, a_{11}\}$ .

Now, let  $X_2 = X_1 \cup \{a_3\} = \{a_3\}$  and  $Y_2 = Y_1 \cup N(v_1) = \{a_2, a_4, a_5, a_6, a_{11}\}$ . Also, note that for the vertices,  $a_4, a_5, a_6 \in N(v_1)$ , we have  $a_4a_5, a_5a_6 \in E(G)$  and hence  $E_2 = \{a_4a_5, a_5a_6\}$ .

Here, label the vertex  $a_3 \in X_2$  by a subset of  $X$ , that is in  $\mathcal{X}$ , and label the vertices of  $Y_2$  by distinct singleton subsets of  $X$  that are in  $\mathcal{S}$ .

Next, reduce the graph  $G$  as  $G_2$  such that  $V(G_2) = V(G_1) - N[v_1]$ . In this reduced graph, the vertex  $a_{12}$  has the maximum degree,  $d(a_{12}) = 5$ . Hence, let  $v_2 = a_{12}$  and  $N(v_2) = \{a_2, a_6, a_9, a_{11}, a_{13}\}$ . Then,  $X_3 = X_2 \cup \{a_{12}\} = \{a_3, a_{12}\}$  and  $Y_3 = Y_2 \cup N(v_2) = \{a_2, a_4, a_5, a_6, a_9, a_{11}, a_{13}\}$ . Also, note that there is no two vertices in  $N(v_2)$  are adjacent in  $G_2$  and hence  $E_3 = E_2$ . Now, label the vertex  $a_{12} \in X_3$  by a subset of  $X$  in  $\mathcal{X}$  and label the unlabeled vertices in  $Y_3$  (that is, the vertices in  $Y_3 - Y_2$ ), by distinct singleton subsets of  $X$  in  $\mathcal{S}$ , which are not used for labeling in previous iterations.

Next, reduce the graph  $G_2$  in to a new graph  $G_3$  such that  $V(G_3) = V(G_2) - N[v_2]$ . In this reduced graph  $G_3$ , the vertex  $a_7$  has the maximum degree,  $d(a_7) = 4$ . Hence, let  $v_3 = a_7$  and then  $N(v_3) = \{a_1, a_2, a_8, a_{11}\}$ . Now, let  $X_4 = X_3 \cup \{a_7\} = \{a_3, a_{12}, a_7\}$  and  $Y_4 = Y_3 \cup N(v_3) = \{a_1, a_2, a_4, a_5, a_6, a_9, a_{11}, a_{13}\}$ . Here, for the vertices,  $a_1, a_2, a_8 \in N(v_3)$ , we have  $a_1a_2, a_1a_8 \in E(G)$  and hence  $E_4 = E_3 \cup \{a_1a_2, a_1a_8\} = \{a_4a_5, a_5a_6, a_1a_2, a_1a_8\}$ . Label the vertex  $a_7 \in X_4$  by a non-singleton set in  $\mathcal{X}$ , which is not used for labeling before and the vertices in  $Y_4 - Y_3$  by singleton sets in  $\mathcal{S}$ , which are not already used.

Next, reduce the graph  $G_3$  in to the graph  $G_4$  such that  $V(G_4) = V(G_3) - N[v_3]$ . In  $G_4$ , the vertex  $a_{10}$  has the maximum degree  $d(a_{10}) = 3$ . Hence, let  $v_4 = a_{10}$  and then  $N(v_4) = \{a_8, a_9, a_{11}\}$ . Now, let  $X_5 = X_4 \cup \{a_{10}\} = \{a_3, a_{12}, a_7, a_{10}\}$  and  $Y_5 = Y_4 \cup N(v_4) = \{a_1, a_2, a_4, a_5, a_6, a_8, a_9, a_{11}, a_{13}\}$ . Here, for the vertices,  $a_8, a_9 \in N(v_4)$ , we have  $a_8a_9 \in E(G)$  and hence  $E_5 = E_4 \cup \{a_8a_9\} = \{a_4a_5, a_5a_6, a_1a_2, a_1a_8, a_8a_9\}$ .

Next, reduce the graph  $G_4$  in to the graph  $G_5$  such that  $V(G_5) = V(G_4) - N[v_4]$ . In  $G_5$ , the vertex  $a_{14}$  has the maximum degree  $d(a_{14}) = 3$ . Hence, let  $v_5 = a_{14}$  and then  $N(v_5) = \{a_4, a_5, a_{13}\}$ . Now, let  $X_5 = X_4 \cup \{a_{14}\} = \{a_3, a_{12}, a_7, a_{10}\}$  and  $Y_5 = Y_4 \cup N(v_4) = \{a_1, a_2, a_4, a_5, a_6, a_8, a_9, a_{11}, a_{13}\}$ . Here, for the vertices,  $a_4, a_5, a_{13} \in N(v_5)$ , we have  $a_4a_5, a_5a_{13} \in E(G)$  and hence  $E_5 = E_4 \cup \{a_4a_5, a_5a_{13}\} = \{a_4a_5, a_5a_6, a_1a_2, a_1a_8, a_8a_9, a_5a_{13}\}$ .

Next, reduce the graph  $G_5$  in to the graph  $G_6$  such that  $V(G_6) = V(G_5) - N[v_5]$ . This reduced graph is a trivial graph. Now, all the vertices of the given graph  $G$  have been labeled by the subsets of  $X$  in such a way that at least one end vertex of every edge in  $G$  has singleton set-label.

Now, label the edges in  $G$  in such a way that the set-label of every edge is the sumset of the set-labels of its end vertices. A labeling of the above graph  $G$  as mentioned in the sparing number algorithm is illustrated in Figure 2.

Note that all edges listed in  $E_5$  will be mono-indexed. Therefore, the sparing number of  $G$  is given by  $\varphi(G) = |E_5| = 6$ .

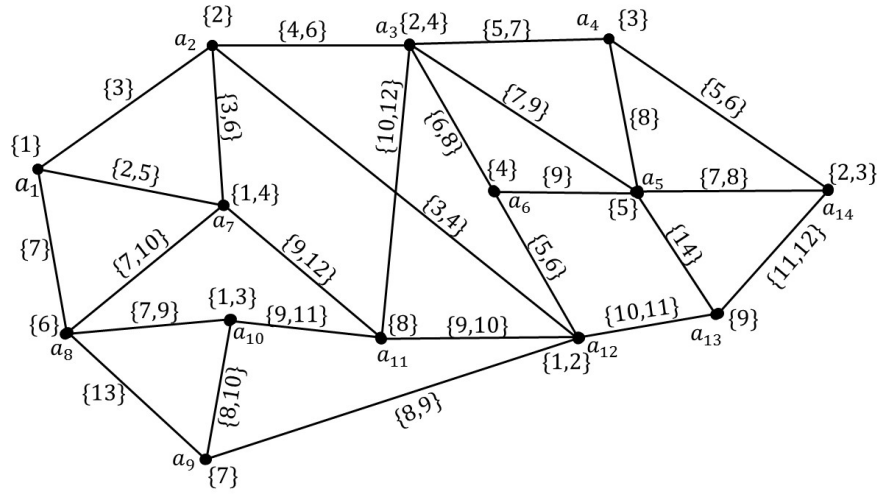


Figure 2

### 3 Conclusion

In this paper, we discussed an algorithm to determine the sparing number of arbitrary WIASL-graphs. A detailed study on the complexity of this algorithm seems to be possible and is interesting for further investigation.

Further studies on many other characteristics of different IASL-graphs are also interesting and challenging. All these facts highlight the scope for further studies in this area.

### 4 Acknowledgements

The authors would like to dedicate this paper to the bright memory of Prof. (Dr.) B. D. Acharya who introduced the concepts of set-valuations of graphs.

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